MATLAB Marina: Numerical Integration

Student Learning Objectives

After completing this module, one should:

1. Be able to use MATLAB to compute numerical integrals.
2. Be able to explain the limitations of numerical integration.

Terms

numerical integration, complete (definite) integral, continuous integral, noisy data

MATLAB Functions, Keywords, and Operators

sum, trapz, cumsum, cumtrapz

Numerical Integration

The definite integral of a function $f(x)$ can be interpreted as an area $R = \int_a^b f(x)\,dx$ and the indefinite integral of a function can be interpreted as the anti-derivative of a function $f(x) = \int \frac{df(x)}{dx}\,dx$. Numerical integration is typically used when a formula for the function to integrate is not available or the integral would be difficult to determine analytically. Numerical integration can also be used to compute estimates of integrals that may be solved analytically later. Numerical integration can yield accurate results if the function is piecewise continuous, relatively smooth, and the increment between function points is relatively small. Numerical integration typically yields better results than numerical differentiation as it is less sensitive to inaccuracies in the data (noisy data).

The Trapezoidal rule and Simpson's rule are two common methods of performing numerical integration for definite integrals. For the Trapezoidal rule, the area between the function and the x-axis is approximated by a trapezoid.

$$\int_a^b f(x)\,dx \approx (b-a)\frac{f(a)+f(b)}{2}$$

For numerical integration using the composite Trapezoidal rule, the interval of integration is split into N small uniform subintervals, each approximated by a trapezoid, and the area of all the trapezoids is summed.

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2N}\left(f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{N-1})+f(x_N)\right)$$

Where the x_k are the end points of the trapezoids

$$x_k = a + k\frac{b-a}{N},\ k = 0,1,\cdots,N$$

For numerical integration using Simpson's rule, the interval is approximated by a quadratic function, and the integral is approximated by the area between the quadratic functions and the x-axis.

$$\int_a^b f(x)\,dx \approx \frac{(b-a)}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)$$

Breaking the interval up into 2N subintervals and approximating each interval by a quadratic function, and summing the areas between the quadratic functions and the x-axis, the integral is approximated by the composite Simpson's rule

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)\cdots+2f(x_{2N-2})+4f(x_{2N-1})+f(x_{2N})\right)$$

Where the x_k are the end points of the trapezoids

$$x_k = a+k\frac{b-a}{2N},\ k=0,1,\cdots,2N$$

and

$$h=\frac{b-a}{2N}$$

Cumulative numerical integration is used when we want to determine an approximation of the anti-derivative of a function.

$$\int_{-\infty}^{t} f(\tau)\,d\tau$$

Numerical Integration using MATLAB

The MATLAB function `trapz` is used to approximate a definite integral of a function using the Trapezoidal rule. The `trapz` function takes two arguments x and y and returns the integral of y with respect to x. The vectors x and y must be the same length and the interval for the approximate definite integral is specified via the x vector values. The result of the `trapz` function will be a scalar value.

The MATLAB functions `cumtrapz` and `cumsum` are used to approximate the anti-derivative of a function (cumulative numerical integral). The `cumtrapz` function takes two arguments x and y and returns the cumulative integral of y with respect to x using the trapezoidal method. The vectors x and y must be the same length, the vector x should have uniform spacing, and the result is a vector the same size as x and y. If the spacing in x is one (distance between x values is one), then `cumtrapz` can be used with a single argument y. The result of the `cumtrapz` function will be an array of the same dimensions as x and y.

The MATLAB program of Figure 1a computes the approximate definite integral of $\int_{-5}^{5}\left(x^2+2x+1\right)dx$ and cumulative integral (approximate anti-derivative) of $\int_{-5}^{x}\left(u^2+2u+1\right)du$.

Figure 1b shows the function and its cumulative integral.

```
% function f(x) = x^2 + 2x + 1
coef = [1, 2, 1];
x = -5 : 0.1 : 5;
f = polyval(coef, x);

% approximate definite integral of f(x) from -5 to 5
defintf = trapz(x,f);
% approximate anti-derivative of f(x)
cumintf = cumtrapz(x,f);

figure(1)
plot(x,f,'b-',x,cumintf,'g-')
xlabel('x')
legend('f(x)','approximate anti-derivative of f(x)')
```

Figure 1a. MATLAB Program Performing Numerical Integration of Polynomial Function

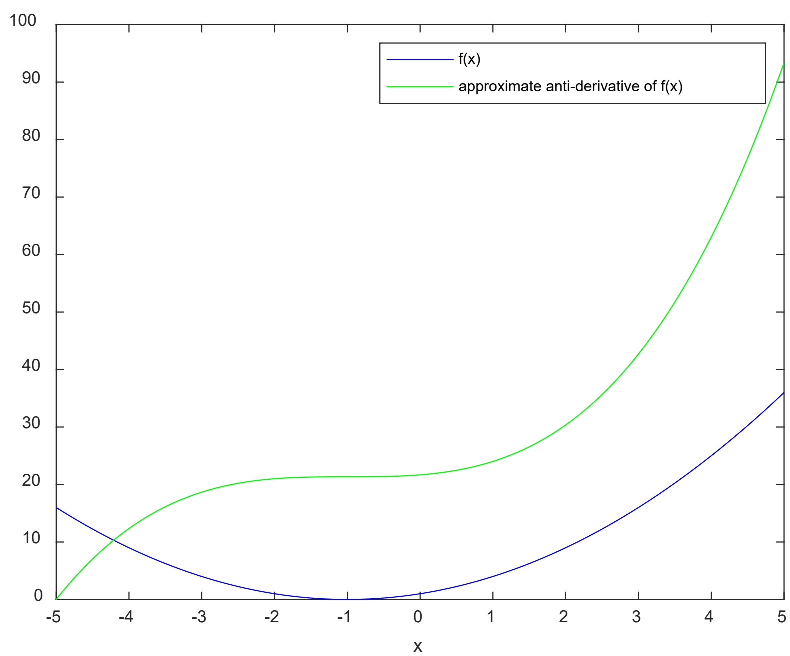

Figure 1b. Approximate Anti-derivative of Polynomial Function

If the spacing in x is not uniform, then `cumtrapz(y)` can be used to compute the cumulative integral with unit spacing. This result must then be element by element multiplied with the spacings. The `cumsum` function can be used similar to `cumtrapz` to perform cumulative numerical integration except it only takes one argument and must either be multiplied by the spacing increment or element by element multiplied with the spacings for non-uniform spacing.

Last modified October 21, 2020